

L_∞ -estimates for the torsion function and L_∞ -growth of semigroups satisfying Gaussian bounds

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Abstract

We investigate selfadjoint C_0 -semigroups on Euclidean domains satisfying Gaussian upper bounds. Major examples are semigroups generated by second order uniformly elliptic operators with Kato potentials and magnetic fields. We study the long time behaviour of the L_∞ operator norm of the semigroup. As an application we prove a new L_∞ -bound for the torsion function of a Euclidean domain that is close to optimal.

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1 Introduction and main results

The following are our standing assumptions.

- (A) Let $\Omega \subseteq \mathbb{R}^d$ be measurable, where $d \in \mathbb{N}$, and let H be a selfadjoint operator in $L_2(\Omega)$ with the following properties:

$$E_0 := E_0(H) := \inf \sigma(H) > -\infty,$$

and the C_0 -semigroup generated by $-H$ satisfies Gaussian upper bounds: e^{-tH} has an integral kernel p_t , for every $t > 0$, and there exist $M \geq 1$, $\omega \in \mathbb{R}$ and $a > 0$ such that

$$|p_t(x, y)| \leq M e^{\omega t} \cdot (a\pi t)^{-d/2} \exp\left(-\frac{|x - y|^2}{at}\right) \quad (1.1)$$

for all $t > 0$ and a.e. $x, y \in \Omega$.

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Expressed differently, (1.1) means that e^{-tH} is dominated by $Me^{\omega t}e^{\frac{a}{4}t\Delta}$,

$$|e^{-tH}f| \leq Me^{\omega t}e^{\frac{a}{4}t\Delta}|f| \quad (t \geq 0, f \in L_2(\Omega)), \quad (1.2)$$

where $\Delta = \Delta_{\mathbb{R}^d}$ denotes the Laplacian on \mathbb{R}^d and $|f|$ is extended by 0 to a function on all of \mathbb{R}^d . Assumption (A) is satisfied, e.g., if Ω is an open set and H is a selfadjoint second order uniformly elliptic operator in divergence form subject to Dirichlet boundary conditions. If some regularity of the boundary of Ω is assumed, then a variety of other boundary conditions are covered, such as Neumann or Robin boundary conditions. Moreover, H may include a magnetic field and a potential from the Kato class.

The first topic of the paper is to investigate the asymptotic behaviour of $\|e^{-tH}\|_{\infty \rightarrow \infty}$, i.e., of the norm of the semigroup operators in $\mathcal{L}(L_\infty(\Omega))$. In [Sim80; formula (1.9)], the estimate

$$\|e^{-tH}\|_{\infty \rightarrow \infty} \leq C(1+t)^{d/2}e^{-E_0t}$$

was shown for a certain class of Schrödinger operators H on \mathbb{R}^d . This estimate was substantially improved and generalized in [Ouh06a; Thm. 4] for operators on Euclidean domains and in [Ouh06b; Thm. 7] for operators on open subsets of complete Riemannian manifolds: If the semigroup generated by $-H$ satisfies Gaussian upper bounds, then

$$\|e^{-tH}\|_{\infty \rightarrow \infty} \leq C(1+t \ln t)^{d/4}e^{-E_0t}.$$

Here we show that the term $\ln t$ can be removed, albeit only for the case of operators on subsets of \mathbb{R}^d . We point out that all the constants in our estimate are explicit.

1.1 Theorem. *Let Assumption (A) hold. Then*

$$\|e^{-tH}\|_{\infty \rightarrow \infty} \leq 2^{1/4}M\left(1 + \frac{5.56}{d}(E_0 + \omega)t\right)^{d/4}e^{-E_0t}$$

for all $t \geq 0$.

1.2 Remark. As pointed out in [Ouh06a], the exponent $d/4$ is sharp in dimension $d = 4$: in [Sim81; Thm. 3.1], examples are given where $E_0 = 0$ and $\|e^{-tH}\|_{\infty \rightarrow \infty}$ grows like $\frac{(1+t)^{4/4}}{\ln t}$. It is not clear if $d/4$ is sharp in other dimensions.

For our second main result we specialize to the case that Assumption (A) is satisfied with $M = 1$, $\omega = 0$ and $a = 4$, i.e., e^{-tH} is dominated by the free heat semigroup on \mathbb{R}^d ,

$$|e^{-tH}f| \leq e^{t\Delta}|f| \quad (t \geq 0, f \in L_2(\Omega)). \quad (1.3)$$

An important example for the operator $-H$ is the Dirichlet Laplacian with magnetic field and a locally integrable absorption potential on an open set $\Omega \subseteq \mathbb{R}^d$. For

more general absorption potentials the space of strong continuity of the semigroup will be $L_2(\Omega')$ for some measurable $\Omega' \subseteq \Omega$.

Assuming $E_0(H) > 0$, we are going to study the quantity

$$q(H) := \frac{\|H^{-1}\|_{\infty \rightarrow \infty}}{\|H^{-1}\|_{2 \rightarrow 2}} = E_0(H) \cdot \|H^{-1}\|_{\infty \rightarrow \infty}. \quad (1.4)$$

Note that $q(H) \geq 1$ by duality and Riesz-Thorin interpolation, and $q(H) = 1$ if $H = -\Delta_{\mathbb{R}^d} + c$ for some $c > 0$.

1.3 Remark. (a) If H^{-1} is a positivity preserving operator, then

$$\|H^{-1}\|_{\infty \rightarrow \infty} = \|H^{-1}1\|_{\infty}$$

so that

$$q(H) = E_0(H) \cdot \|H^{-1}1\|_{\infty} \quad (1.5)$$

in this case.

(b) Let D be an open subset of \mathbb{R}^d and $H = -\Delta_D$, where Δ_D denotes the Dirichlet Laplacian on D . Then $u_D := H^{-1}1$ is the torsion function of D , and by (1.5) we have

$$\|u_D\|_{\infty} = \frac{q(H)}{E_0(H)}.$$

Thus, the bounds for the quantity $q(H)$ that we prove in Theorem 1.5 below lead to bounds for the L_{∞} -norm of the torsion function u_D .

We first consider the case that $H = -\Delta_{B_d}$, where B_d denotes the open unit ball in \mathbb{R}^d .

1.4 Lemma. *There exists $C > 0$ such that*

$$\frac{d}{8} \leq q(-\Delta_{B_d}) \leq \frac{d}{8} + Cd^{1/3}$$

for all dimensions d .

It is easy to see that $q(-\Delta_B) = q(-\Delta_{B_d})$ for all balls $B \subset \mathbb{R}^d$. In [Ber12; top of p. 614], M. van den Berg conjectured that $q(-\Delta_D) \leq q(-\Delta_{B_d})$ for all bounded open subsets $D \subset \mathbb{R}^d$. Our second main result gives some indication that this conjecture might be true even for the general operator H in place of $-\Delta_D$; note that our estimate below with $c\sqrt{d}$ is only slightly worse than the estimate with $Cd^{1/3}$ in Lemma 1.4.

1.5 Theorem. *Let Assumption (A) hold with the stronger estimate (1.3). Then*

$$1 \leq q(H) \leq \frac{d}{8} + c\sqrt{d} + 1,$$

where $c := \frac{1}{4}\sqrt{5(1 + \frac{1}{4}\ln 2)} < 0.61$.

1.6 Remark. (a) In [BeCa09; Thm. 1] it is shown that

$$q(-\Delta_D) \leq 3 \ln 2 \cdot d + 4$$

for all open subsets $D \subset \mathbb{R}^d$ with $E_0(-\Delta_D) > 0$. Theorem 1.5 improves the constant $3 \ln 2$ in front of d to the optimal $\frac{1}{8}$ (cf. Lemma 1.4).

(b) Clearly, the upper bound $C_d := \frac{d}{8} + c\sqrt{d} + 1$ in Theorem 1.5 is not optimal. But, as discussed above, $\frac{d}{8}$ is the correct leading order term in high dimensions. It turns out that the bound C_d is also quite good in low dimensions. Here is a table of approximate values for $q_d := q(-\Delta_{B_d})$ and C_d :

d	1	2	3	4	5
q_d	1.2337	1.4458	1.6449	1.8352	2.0191
C_d	1.7305	2.1063	2.4238	2.7110	2.9790

It appears that C_d is never off by more than a factor of 1.5.

The proofs of Theorems 1.1 and 1.5 will be given in Section 3. In Section 2 we provide the necessary tools and prove an auxiliary result.

2 The method of weighted estimates

The aim of this section is to prove the following theorem, which will be used in the proofs of our two main results.

2.1 Theorem. *Let Assumption (A) hold. Then for all $\varepsilon \in (0, 1]$ one has*

$$\|e^{-tH}\|_{\infty \rightarrow \infty} \leq 2^{1/4} M \left(\frac{1 + 1/\sqrt{\varepsilon}}{2} \right)^{d/2} e^{\varepsilon(E_0 + \omega)t - E_0 t} \quad (t \geq 0).$$

2.2 Remark. The precise form of the factor $\left(\frac{1 + 1/\sqrt{\varepsilon}}{2} \right)^{d/2}$ will be important for the proof of Theorem 1.5, where $\omega = 0$: given $t > 0$, it is not difficult to show that there exists $\varepsilon \in (0, 1]$ with

$$\left(\frac{1 + 1/\sqrt{\varepsilon}}{2} \right)^{d/2} e^{\varepsilon E_0 t - E_0 t} < 1$$

if and only if $E_0 t > \frac{d}{8}$. This is the origin of the leading term $\frac{d}{8}$ in the upper estimate of Theorem 1.5.

The proof of Theorem 2.1 is based on the method of weighted estimates. We need two ingredients: a result on complex interpolation and a way to derive $L_\infty \rightarrow L_\infty$ -estimates from weighted $L_2 \rightarrow L_\infty$ -estimates. Our first ingredient is a refinement of Proposition 3.1 from [Vog15], where the case $\varepsilon = \omega = 0$ is proved. We denote $\mathbb{C}_+ := \{z \in \mathbb{C}; \operatorname{Re} z > 0\}$.

2.3 Proposition. *Let (Ω, μ) be a measure space, and let $\rho: \Omega \rightarrow \mathbb{R}$ be measurable. Let $E_0 \in \mathbb{R}$, and let $T: \mathbb{C}_+ \rightarrow \mathcal{L}(L_2(\mu))$ be analytic, $\|T(z)\|_{2 \rightarrow 2} \leq e^{-E_0 \operatorname{Re} z}$ for all $z \in \mathbb{C}_+$. Assume that for every $\varepsilon > 0$ there exist $C \geq 0$ and $\omega \in \mathbb{R}$ such that*

$$\|e^{-\alpha\rho}T(t)e^{\alpha\rho}\|_{2 \rightarrow 2} \leq Ce^{(1+\varepsilon)\alpha^2 t + \omega t} \quad (\alpha, t > 0).$$

Then $\|e^{-\alpha\rho}T(t)e^{\alpha\rho}\|_{2 \rightarrow 2} \leq e^{\alpha^2 t - E_0 t}$ for all $\alpha, t > 0$.

Here and in the following we denote

$$\|e^{-\rho}B e^{\rho}\|_{p \rightarrow q} := \sup\{\|e^{-\rho}B e^{\rho}f\|_q; f \in L_p(\mu), \|f\|_p \leq 1, e^{\rho}f \in L_2(\mu)\}$$

for a given operator $B \in \mathcal{L}(L_2(\mu))$, a measurable function $\rho: \Omega \rightarrow \mathbb{C}$ and $p, q \in [1, \infty]$.

Proof of Proposition 2.3. We define a rescaled analytic function $\tilde{T}: \mathbb{C}_+ \rightarrow \mathcal{L}(L_2(\mu))$ by

$$\tilde{T}(z) := \exp\left(-\frac{\omega z}{1+\varepsilon}\right)T\left(\frac{z}{1+\varepsilon}\right).$$

Then

$$\|\tilde{T}(z)\|_{2 \rightarrow 2} \leq \exp\left(-\frac{\omega \operatorname{Re} z}{1+\varepsilon} - E_0 \frac{\operatorname{Re} z}{1+\varepsilon}\right) = \exp\left(-\frac{\omega + E_0}{1+\varepsilon} \operatorname{Re} z\right)$$

for all $z \in \mathbb{C}_+$. Moreover,

$$\|e^{-\alpha\rho}\tilde{T}(t)e^{\alpha\rho}\|_{2 \rightarrow 2} \leq \exp\left(-\frac{\omega t}{1+\varepsilon}\right) \cdot C \exp\left((1+\varepsilon)\alpha^2 \frac{t}{1+\varepsilon} + \omega \frac{t}{1+\varepsilon}\right) = C \exp(\alpha^2 t)$$

for all $t > 0$. Thus we can apply [Vog15; Prop. 3.1] to obtain

$$\|e^{-\alpha\rho}\tilde{T}(t)e^{\alpha\rho}\|_{2 \rightarrow 2} \leq \exp\left(\alpha^2 t - \frac{\omega + E_0}{1+\varepsilon} t\right)$$

for all $t > 0$ and hence

$$e^{-\omega t}\|e^{-\alpha\rho}T(t)e^{\alpha\rho}\|_{2 \rightarrow 2} = \|e^{-\alpha\rho}\tilde{T}((1+\varepsilon)t)e^{\alpha\rho}\|_{2 \rightarrow 2} \leq e^{\alpha^2(1+\varepsilon)t - (\omega + E_0)t}.$$

Multiplying by $e^{\omega t}$ and letting $\varepsilon \rightarrow 0$ we obtain the asserted estimate. \square

We will work with the weight functions $e^{\pm\alpha\rho_w}$, where $\alpha > 0$ and $\rho_w: \mathbb{R}^d \rightarrow [0, \infty]$ is defined by

$$\rho_w(x) := |x - w|,$$

for given $w \in \mathbb{R}^d$. Note that $e^{-\alpha\rho_w} \in L_1 \cap L_\infty$. We will need a good estimate for the integral of $e^{-\alpha\rho_w}$.

2.4 Lemma. (a) *For $x > 0$ one has $\Gamma(x + \frac{1}{2}) \leq (\frac{x}{e})^x \sqrt{2\pi}$.*

(b) *For $\alpha > 0$ one has $\int_{\mathbb{R}^d} e^{-\alpha|y|} dy \leq \sqrt{2} \left(\frac{2\pi d}{e}\right)^{d/2} \alpha^{-d}$.*

Proof. (a) We have to show that

$$f(x) := x \ln x - x + \ln \sqrt{2\pi} - \ln \Gamma(x + \tfrac{1}{2}) \geq 0$$

for all $x > 0$. More strongly, we show that

$$f(x) = \int_0^\infty \frac{1}{t} \left(\frac{1}{t} - \frac{1}{2 \sinh \frac{t}{2}} \right) e^{-tx} dt =: g(x)$$

for all $x > 0$, which even implies that f is completely monotone.

First observe that the function $t \mapsto \frac{1}{t} \left(\frac{1}{t} - \frac{1}{2 \sinh(t/2)} \right)$ is bounded on $(0, \infty)$, so g is defined and $g(x) \rightarrow 0$ as $x \rightarrow \infty$. Moreover, by Stirling's formula we obtain

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \left(x \ln x - x - x \ln(x + \tfrac{1}{2}) + x + \tfrac{1}{2} \right) = \lim_{x \rightarrow \infty} \left(x \ln \frac{x}{x+1/2} + \tfrac{1}{2} \right) = 0.$$

According to [AbSt72; 6.4.1] we have $(\ln \Gamma)''(x) = \int_0^\infty \frac{t}{1-e^{-t}} e^{-xt} dt$ for all $x > 0$. It follows that

$$\begin{aligned} f''(x) &= \frac{1}{x} - (\ln \Gamma)''(x + \tfrac{1}{2}) = \int_0^\infty \left(1 - \frac{te^{-t/2}}{1-e^{-t}} \right) e^{-xt} dt \\ &= \int_0^\infty \frac{1}{t} \left(\frac{1}{t} - \frac{1}{e^{t/2} - e^{-t/2}} \right) t^2 e^{-xt} dt = g''(x) \end{aligned}$$

for all $x > 0$. Together with $\lim_{x \rightarrow \infty} (f - g)(x) = 0$ we conclude that $f - g = 0$.

(b) Assume without loss of generality that $\alpha = 1$. With σ_{d-1} denoting the surface measure of the unit sphere we compute

$$\int_{\mathbb{R}^d} e^{-|y|} dy = \sigma_{d-1} \int_0^\infty r^{d-1} e^{-r} dr = \frac{2\pi^{d/2}}{\Gamma(d/2)} \Gamma(d).$$

By [AbSt72; 6.1.18] and part (a) we obtain

$$\Gamma(2x)/\Gamma(x) = (2\pi)^{-1/2} 2^{2x-1/2} \Gamma(x + \tfrac{1}{2}) \leq 2^{2x-1/2} \left(\frac{x}{e}\right)^x$$

for all $x > 0$. It follows that

$$\int_{\mathbb{R}^d} e^{-|y|} dy \leq 2\pi^{d/2} \cdot 2^{d-1/2} \left(\frac{d}{2e}\right)^{d/2} = \sqrt{2} \left(\frac{2\pi d}{e}\right)^{d/2}. \quad \square$$

The following is our second ingredient in the method of weighted estimates.

2.5 Proposition. *Let $\Omega \subseteq \mathbb{R}^d$ be measurable, $\alpha > 0$, and let B be a bounded operator on $L_2(\Omega)$ satisfying*

$$\|e^{-\alpha \rho_w} B e^{\alpha \rho_w}\|_{2 \rightarrow \infty} \leq 1$$

for all $w \in \Omega$. Then

$$\|B\|_{\infty \rightarrow \infty} \leq 2^{1/4} \left(\frac{\pi d}{2e}\right)^{d/4} \alpha^{-d/2}.$$

Proof. Let $f \in L_2(\Omega) \cap L_\infty(\Omega)$ have bounded support. Observe that $\|Bf\|_\infty = \sup_{w \in \Omega} \|e^{-\alpha\rho_w} Bf\|_\infty$. For all $w \in \Omega$ we can use the assumption to estimate

$$\|e^{-\alpha\rho_w} Bf\|_\infty \leq \|e^{-\alpha\rho_w} f\|_2 \leq \|e^{-\alpha\rho_w}\|_2 \|f\|_\infty.$$

By Lemma 2.4 we have

$$\|e^{-\alpha\rho_w}\|_2^2 = \int_{\Omega} e^{-2\alpha|y|} dy \leq \sqrt{2} \left(\frac{\pi d}{2e}\right)^{d/2} \alpha^{-d},$$

so it follows that $\|Bf\|_\infty \leq 2^{1/4} \left(\frac{\pi d}{2e}\right)^{d/4} \alpha^{-d/2} \|f\|_\infty$. A simple approximation shows that the same estimate holds for arbitrary $f \in L_2(\Omega) \cap L_\infty(\Omega)$, which proves the assertion. \square

In order to apply Proposition 2.5 in the proof of Theorem 2.1, we need the following weighted estimates of the free heat semigroup $(e^{t\Delta})_{t \geq 0}$ on \mathbb{R}^d .

2.6 Lemma. *Let $w \in \mathbb{R}^d$ and $\alpha > 0$. Then*

$$\|e^{-\alpha\rho_w} e^{t\Delta} e^{\alpha\rho_w}\|_{2 \rightarrow 2} \leq e^{\alpha^2 t}, \quad \|e^{-\alpha\rho_w} e^{t\Delta} e^{\alpha\rho_w}\|_{2 \rightarrow \infty} \leq \left(1 + \frac{1}{\beta}\right)^{\frac{d}{4}} (8\pi t)^{-d/4} e^{(1+\beta)\alpha^2 t}$$

for all $t, \beta > 0$.

Proof. Let $x, y, w \in \mathbb{R}^d$ and $\alpha, \beta, t > 0$. Let k_t be the convolution kernel of $e^{s\Delta}$. First observe that

$$-\alpha|x-w| + \alpha|y-w| \leq \alpha|x-y| \leq (1+\beta)t\alpha^2 + \frac{|x-y|^2}{4(1+\beta)t}.$$

Since $\frac{1}{4(1+\beta)t} - \frac{1}{4t} = -\frac{\beta}{4(1+\beta)t}$, it follows that

$$\begin{aligned} e^{-\alpha|x-w|} k_t(x-y) e^{\alpha|y-w|} &\leq (4\pi t)^{-d/2} \exp\left((1+\beta)\alpha^2 t - \frac{\beta}{1+\beta} \frac{|x-y|^2}{4t}\right) \\ &= \left(\frac{1+\beta}{\beta}\right)^{d/2} e^{(1+\beta)\alpha^2 t} k_s(x-y), \end{aligned}$$

with $s = \frac{1+\beta}{\beta}t$. Therefore,

$$\|e^{-\alpha\rho_w} e^{t\Delta} e^{\alpha\rho_w}\|_{2 \rightarrow 2} \leq \left(\frac{1+\beta}{\beta}\right)^{d/2} e^{(1+\beta)\alpha^2 t} \|e^{s\Delta}\|_{2 \rightarrow 2}.$$

Since $\|e^{z\Delta}\|_{2 \rightarrow 2} \leq 1$ for all $z \in \mathbb{C}_+$, the function $z \mapsto e^{z\Delta}$ satisfies the assumptions of Proposition 2.3 with $E_0 = 0$, and the first assertion follows.

Similarly,

$$\|e^{-\alpha\rho_w} e^{t\Delta} e^{\alpha\rho_w}\|_{2 \rightarrow \infty} \leq \left(\frac{1+\beta}{\beta}\right)^{d/2} e^{(1+\beta)\alpha^2 t} \|e^{s\Delta}\|_{2 \rightarrow \infty}.$$

This implies the second assertion since $\|e^{s\Delta}\|_{2 \rightarrow \infty} = (8\pi s)^{-d/4} = \left(\frac{1+\beta}{\beta}\right)^{-d/4} (8\pi t)^{-d/4}$. \square

Now we are ready to prove the main result of this section.

Proof of Theorem 2.1. We first show the assertion in the case where $E_0 = 0$ and $a = 4$; then the general assertion is proved by rescaling.

Assumption (A) implies $\|e^{-zH}\|_{2 \rightarrow 2} \leq e^{-E_0 \operatorname{Re} z} = 1$ for all $z \in \mathbb{C}_+$, and

$$\|e^{-\alpha\rho_w} e^{-tH} e^{\alpha\rho_w}\|_{2 \rightarrow 2} \leq M e^{\omega t} \|e^{-\alpha\rho_w} e^{t\Delta} e^{\alpha\rho_w}\|_{2 \rightarrow 2} \leq M e^{\omega t + \alpha^2 t} \quad (w \in \mathbb{R}^d, \alpha, t > 0)$$

by Lemma 2.6. By Proposition 2.3 it follows that

$$\|e^{-\alpha\rho_w} e^{-tH} e^{\alpha\rho_w}\|_{2 \rightarrow 2} \leq e^{\alpha^2 t} \quad (w \in \mathbb{R}^d, \alpha, t > 0). \quad (2.1)$$

Due to (1.2) and Lemma 2.6 we have

$$\begin{aligned} \|e^{-\alpha\rho_w} e^{-tH} e^{\alpha\rho_w}\|_{2 \rightarrow \infty} &\leq M e^{\omega t} \|e^{-\alpha\rho_w} e^{t\Delta} e^{\alpha\rho_w}\|_{2 \rightarrow \infty} \\ &\leq M (8\pi t)^{-d/4} \left(1 + \frac{1}{\beta}\right)^{d/4} e^{\omega t + (1+\beta)\alpha^2 t}. \end{aligned}$$

Using the semigroup property and (2.1), we deduce for $\varepsilon \in (0, 1]$ that

$$\begin{aligned} \|e^{-\alpha\rho_w} e^{-tH} e^{\alpha\rho_w}\|_{2 \rightarrow \infty} &\leq \|e^{-\alpha\rho_w} e^{-\varepsilon t H} e^{\alpha\rho_w}\|_{2 \rightarrow \infty} \|e^{-\alpha\rho_w} e^{-(1-\varepsilon)tH} e^{\alpha\rho_w}\|_{2 \rightarrow 2} \\ &\leq M (8\pi \varepsilon t)^{-d/4} \left(1 + \frac{1}{\beta}\right)^{d/4} e^{\omega \varepsilon t + (1+\beta)\alpha^2 \varepsilon t + \alpha^2 (1-\varepsilon)t}. \end{aligned}$$

By Proposition 2.5 it follows that

$$\begin{aligned} \|e^{-tH}\|_{\infty \rightarrow \infty} &\leq 2^{1/4} \left(\frac{\pi d}{2\varepsilon}\right)^{d/4} \alpha^{-d/2} \cdot M (8\pi \varepsilon t)^{-d/4} \left(1 + \frac{1}{\beta}\right)^{d/4} e^{\omega \varepsilon t + (1+\beta\varepsilon)\alpha^2 t} \\ &\leq 2^{1/4} M \left(\frac{d}{16\varepsilon}\right)^{d/4} \cdot (\alpha^2 t)^{-d/4} \left(1 + \frac{1}{\beta}\right)^{d/4} e^{\omega \varepsilon t + (1+\beta\varepsilon)\alpha^2 t}. \end{aligned}$$

The right hand side in the previous inequality becomes minimal for $\alpha^2 = \frac{d/4}{(1+\beta\varepsilon)t}$. Then $(\alpha^2 t)^{-d/4} = \left(\frac{4}{d}(1+\beta\varepsilon)\right)^{d/4}$, so

$$\|e^{-tH}\|_{\infty \rightarrow \infty} \leq 2^{1/4} M \left(\frac{1+\beta\varepsilon}{4\varepsilon} \left(1 + \frac{1}{\beta}\right)\right)^{d/4} e^{\omega \varepsilon t}.$$

Now the right hand side becomes minimal for $\beta = \varepsilon^{-1/2}$. Then $\frac{1+\beta\varepsilon}{4\varepsilon} \left(1 + \frac{1}{\beta}\right) = \frac{(1+\sqrt{\varepsilon})^2}{4\varepsilon}$, and we conclude that

$$\|e^{-tH}\|_{\infty \rightarrow \infty} \leq 2^{1/4} M \left(\frac{1+1/\sqrt{\varepsilon}}{2}\right)^{d/2} e^{\omega \varepsilon t}.$$

Now we prove the assertion for general $E_0 \in \mathbb{R}$ and $a > 0$. Observe that the operator $\tilde{H} := \frac{4}{a}(H - E_0)$ satisfies Assumption (A) with $\tilde{E}_0 = 0$, $\tilde{\omega} = \frac{4}{a}(\omega + E_0)$ and $\tilde{a} = 4$. Thus we can apply the above to obtain

$$\|e^{-\frac{a}{4}t\tilde{H}}\|_{\infty \rightarrow \infty} \leq 2^{1/4} M \left(\frac{1+1/\sqrt{\varepsilon}}{2}\right)^{d/2} e^{\tilde{\omega}\varepsilon\frac{a}{4}t}$$

for all $t \geq 0$. The assertion in the general case follows since $\|e^{-\frac{a}{4}t\tilde{H}}\|_{\infty \rightarrow \infty} = e^{E_0 t} \|e^{-tH}\|_{\infty \rightarrow \infty}$ and $\tilde{\omega}\varepsilon\frac{a}{4}t = \varepsilon(\omega + E_0)t$. \square

We conclude this section by explaining how sharp the method of weighted estimates from Proposition 2.5 is in certain cases.

2.7 Remark. Fix $t > 0$ and consider $B = e^{t\Delta}$. Applying Lemma 2.6 with $\beta = 1$ gives

$$\|e^{-\alpha\rho_w} e^{t\Delta} e^{\alpha\rho_w}\|_{2\rightarrow\infty} \leq (4\pi t)^{-d/4} e^{2\alpha^2 t}$$

for all $\alpha > 0$ and $w \in \mathbb{R}^d$. We only need this estimate for $\alpha = (\frac{d}{8t})^{1/2}$. Then by Proposition 2.5 we obtain

$$\|e^{t\Delta}\|_{\infty\rightarrow\infty} \leq 2^{1/4} \left(\frac{\pi d}{2e}\right)^{d/4} \alpha^{-d/2} \cdot (4\pi t)^{-d/4} e^{2\alpha^2 t} = 2^{1/4} \left(\frac{d}{8e\alpha^2 t}\right)^{d/4} e^{2\alpha^2 t} = 2^{1/4},$$

which is quite sharp since $\|e^{t\Delta}\|_{\infty\rightarrow\infty} = 1$.

3 Proof of Theorems 1.1 and 1.5

We first prove bounds for $q(-\Delta_{B_d})$, where B_d is the unit ball in \mathbb{R}^d .

Proof of Lemma 1.4. It is easy to show that $(-\Delta_{B_d})^{-1}1(x) = \frac{1}{2d}(1-x^2)$ for all $x \in B_d$, so $\|\Delta_{B_d}^{-1}1\|_{\infty} = \frac{1}{2d}$. In [FMPP07; Example 5.8] it is shown that $E_0(-\Delta_{B_d}) \geq \frac{1}{4}d^2$. Now the lower estimate follows from (1.5):

$$q(-\Delta_{B_d}) = E_0(-\Delta_{B_d}) \cdot \frac{1}{2d} \geq \frac{d}{8}.$$

It is well-known that $E_0(-\Delta_{B_d}) = j_{\frac{d}{2}-1,1}^2$, where $j_{\nu,1}$ denotes the first positive zero of the Bessel function J_{ν} . By [Tri49] it follows that $E_0(-\Delta_{B_d}) = \frac{1}{4}d^2 + O(d^{4/3})$. Thus, $q(-\Delta_B) = \frac{d}{8} + O(d^{1/3})$, which implies the upper estimate. \square

Theorem 1.1 follows from Theorem 2.1 by an optimization with respect to ε .

Proof of Theorem 1.1. Let $t > 0$. By Theorem 2.1 we know that

$$\|e^{-tH}\|_{\infty\rightarrow\infty} \leq 2^{1/4} M \left(\frac{1+1/\sqrt{\varepsilon}}{2}\right)^{d/2} e^{\varepsilon(E_0+\omega)t-E_0 t}$$

for all $\varepsilon \in (0, 1]$. Thus it remains to show that there exists $\varepsilon \in (0, 1]$ such that

$$\left(\frac{1+1/\sqrt{\varepsilon}}{2}\right)^{d/2} e^{\varepsilon(E_0+\omega)t} \leq \left(1 + \frac{5.56}{d}(E_0 + \omega)t\right)^{d/4}. \quad (3.1)$$

Setting $x = (E_0 + \omega)t/d$ and raising both sides to the power $\frac{4}{d}$, we see that (3.1) is equivalent to

$$\left(\frac{1+1/\sqrt{\varepsilon}}{2}\right)^2 e^{4\varepsilon x} \leq 1 + 5.56x.$$

Case 1: $x \leq \alpha := 0.14$. Then we choose $\varepsilon = 1$ and use the inequality

$$e^{4x} \leq 1 + \frac{e^{4\alpha}-1}{\alpha}x \quad (0 \leq x \leq \alpha),$$

which is valid due to the convexity of $x \mapsto e^{4x}$. Now the assertion follows since $\frac{e^{4\alpha}-1}{\alpha} < 5.4$.

Case 2: $x > \alpha$. Set $\tau = 4e^{-4\alpha} - 1 (> 0)$ and choose $\varepsilon = \frac{\alpha}{x} (< 1)$. Then

$$\begin{aligned} \left(\frac{1+1/\sqrt{\varepsilon}}{2}\right)^2 e^{4\varepsilon x} &= \left(1 + \frac{2}{\sqrt{\varepsilon}} + \frac{1}{\varepsilon}\right) \frac{1}{4e^{-4\alpha}} \\ &\leq \left(1 + \tau + \frac{1}{\tau\varepsilon} + \frac{1}{\varepsilon}\right) \frac{1}{\tau+1} = 1 + \frac{1}{\tau\varepsilon} = 1 + \frac{1}{\tau\alpha}x, \end{aligned}$$

and the assertion follows since $\frac{1}{\tau\alpha} < 5.56$. \square

The proof of Theorem 1.5 is also based on an optimization with respect to ε ; however, the required estimations are more involved.

Proof of Theorem 1.5. Recall that the lower bound $q(H) \geq 1$ holds by duality and interpolation.

Let $\varepsilon \in (0, 1)$, and choose $t_0 > 0$ such that

$$2^{1/4} \left(\frac{1+1/\sqrt{\varepsilon}}{2}\right)^{d/2} e^{-(1-\varepsilon)E_0 t_0} = 1. \quad (3.2)$$

The assumption on H implies $\|e^{-tH}\|_{\infty \rightarrow \infty} \leq \|e^{t\Delta}\|_{\infty \rightarrow \infty} = 1$ for all $t \geq 0$. Then by the resolvent formula and Theorem 2.1 we obtain

$$\begin{aligned} \|H^{-1}\|_{\infty \rightarrow \infty} &\leq \int_0^\infty \|e^{-tH}\|_{\infty \rightarrow \infty} dt \\ &\leq t_0 + \int_{t_0}^\infty 2^{1/4} \left(\frac{1+1/\sqrt{\varepsilon}}{2}\right)^{d/2} e^{-(1-\varepsilon)E_0 t} dt = t_0 + \frac{1}{(1-\varepsilon)E_0}. \end{aligned}$$

Thus,

$$q(H) = E_0 \|H^{-1}\|_{\infty \rightarrow \infty} \leq E_0 t_0 + \frac{1}{(1-\varepsilon)}.$$

To prove the upper bound, we now show that

$$(1-\varepsilon)E_0 t_0 + 1 \leq (1-\varepsilon)\left(\frac{d}{8} + c\sqrt{d} + 1\right) \quad (3.3)$$

for a suitable choice of ε .

Set $\gamma := \frac{8}{5}c$; then $\gamma < 1$, $c = \frac{5}{8}\gamma$ and $1 + \frac{1}{4}\ln 2 = \frac{16}{5}c^2 = \frac{5}{4}\gamma^2$. Further set $x := \frac{\gamma}{\sqrt{d}} (< 1)$ and choose $\varepsilon := \frac{1}{(1+2x)^2}$. Then $\varepsilon \in (0, 1)$ as required. By the choice of t_0 in (3.2), the left hand side in (3.3) equals

$$\frac{1}{4}\ln 2 + \frac{d}{2}\ln \frac{1+1/\sqrt{\varepsilon}}{2} + 1 = \frac{5}{4}\gamma^2 + \frac{d}{2}\ln \frac{1+(1+2x)}{2} = \frac{5}{4}dx^2 + \frac{d}{2}\ln(1+x),$$

whereas the right hand side equals

$$\left(1 - \frac{1}{(1+2x)^2}\right) \cdot \frac{d}{8} \left(1 + \frac{8c}{\sqrt{d}} + \frac{8}{d}\right) = \frac{x+x^2}{(1+2x)^2} \cdot \frac{d}{2} \left(1 + 5x + \frac{8}{\gamma^2}x^2\right),$$

so that it remains to prove the inequality

$$\frac{5}{4}dx^2 + \frac{d}{2}\ln(1+x) \leq \frac{x+x^2}{(1+2x)^2} \cdot \frac{d}{2} \left(1 + 5x + \frac{8}{\gamma^2}x^2\right).$$

Since $\frac{8}{\gamma^2} = 10/(1 + \frac{1}{4}\ln 2) \approx 8.523 > 8.5$, it suffices to show

$$\frac{5}{2}x^2 + \ln(1+x) \leq \frac{x+x^2}{(1+2x)^2} \cdot (1 + 5x + 8.5x^2) =: g(x)$$

for all $x \in [0, 1]$. A straightforward computation yields

$$g(x) - \frac{5}{2}x^2 = x - \frac{x^2}{2} + \frac{x^3}{2} \cdot \frac{3+x}{(1+2x)^2} \geq x - \frac{x^2}{2} + \frac{x^3}{2} \cdot \frac{4/5}{1+x}.$$

(The last inequality simplifies to $11 + 4x \geq 11x^2$, which is trivial for $x \in [0, 1]$.)

Thus the assertion follows if

$$f(x) := x - \frac{x^2}{2} + \frac{x^3}{2} \cdot \frac{4/5}{1+x} - \ln(1+x) \geq 0.$$

Note that $f(0) = 0$; we conclude the proof by showing $f' \geq 0$ on $[0, 1]$:

$$\begin{aligned} f'(x) &= 1 - x + \frac{6}{5} \cdot \frac{x^2}{1+x} - \frac{2}{5} \cdot \frac{x^3}{(1+x)^2} - \frac{1}{1+x} \\ &= \frac{1}{1+x} \left(1 - x^2 + \frac{6}{5} \cdot x^2 - \frac{2}{5} \cdot \frac{x^3}{1+x} - 1 \right) \\ &= \frac{1}{5(1+x)} \left(x^2 - \frac{2x^3}{1+x} \right) = \frac{1}{5(1+x)^2} (x^2 - x^3) \geq 0 \end{aligned}$$

for all $x \in [0, 1]$. □

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